ISI – Bangalore Center – B Math - Physics I –End Semester Exam
Date: 28 April 2017. Duration of Exam: 3 hours
Total marks: 100

Answer questions 1-4 and either question 5 or question 6.

Q1. [Total Marks:3+3+6+7+3=22]

A particle P of mass M is suspended from a fixed point O by a light inextensible string of length a. A second particle Q of mass m is in turn suspended from P by another light inextensible string of the same length. The system moves in a vertical plane through O. Let θ and ϕ be the angles that P and Q makes with the verticals respectively.

1a.) Show that the near equilibrium the kinetic energy of the system for small oscillations can be approximated as $T = \frac{1}{2}(M+m)a\dot{\theta}^2 + ma\dot{\phi}\dot{\theta} + \frac{1}{2}ma\dot{\phi}^2$

1b.) Show that to the same approximation the potential energy can be written (upto a constant term) near the equilibrium as $V = \frac{1}{2}(M+m)ga\theta^2 + \frac{1}{2}mga\phi^2$

For the rest of this question take $M = 3m$.

1c.) Show that the equations of motion can be written as

$$3\ddot{\theta} + 4n^2\theta - n^2\phi = 0$$

$$\ddot{\theta} + \ddot{\phi} + n^2\phi = 0$$

where $n^2 = g/a$.

1d.) Find the normal frequencies and the forms of the normal modes for this system.

1e.) Describe the complete solution if initially P is at rest with $\theta = 0$ and Q is at rest with $\phi = \alpha$

Q2. [Total Marks:6+6+5+5=22]

A rigid uniform rod of length l and mass M is hanging by a frictionless fixed pivot from the ceiling. An object of mass m and speed v_0 strikes the hanging rod at right angle at the free end of the hanging rod. Immediately after striking the rod, the object continues forward but its speed decreases to $\frac{v_0}{2}$ and the rod swings in a vertical plane.

2a.) From dimensional analysis the angular speed of the rod immediately after the collision can be written as $\omega = k\frac{mv_0}{Ml}$ where k is constant. Determine the value of k.

2b.) If after the collision the rod can just touch the ceiling on its first swing , then show that $v_0 = \frac{M}{m}\sqrt{\frac{4gl}{3}}$.

2c.) For what value of $\frac{M}{m}$ will the collision be elastic?

Useful information: The moment of inertia of the rod about its center of mass is $\frac{1}{12}Ml^2$. Gravity acts with acceleration g downward.

2d.) Suppose that in the above example, the rigid rod comes off the ceiling just as it gets hit by the moving body. Will the rod stay in the same vertical plane as it falls to the ground ? Assume that the ball and the moving body does not collide again after the initial collision. Explain your answer.

Q3. [Total Marks: 4+6+6+3+3=22]

Consider the system shown in Figure 1. Two particles of mass *m* each are connected by a rigid rod which is of length a and has negligible mass. One of the particles can only slide freely along a thin frictionless horizontal rod and the other can swing freely in a vertical plane under the action of gravity.

Set up the coordinate in such a way that x represent the horizontal displacement of the sliding mass, and θ represents angular displacement of the swinging mass as shown in the picture. Two particles are initially held at rest horizontally and then let go. Choose initial values of x and θ to be zero. Let the initial potential energy be set to zero.

3a.) Show that the constraints of the motion are holonomic in nature.
3b.) Identify the forces of constraints and show that they do not do any work.
3c.) Show that the Lagrangian depends on $\theta, \dot{x}, \dot{\theta}$ but not on x and find that associated conservation law.
3d.) Show that x and θ are related by $x = \frac{a(1-\cos\theta)}{2}$
3e.) The conservation law found in part c.) is related to which of the following: Conservation of linear momentum, conservation of angular momentum, or conservation of energy? Explain your answer.

Q4. [Total Marks: 6+2+5+2+4+3=22]

Suppose that a rigid body is moving with a point in the rigid body remaining fixed. Take the fixed point as the origin of coordinate axis.

Remember that the moment of interia matrix is defined as $I_{ij} = \sum_{\alpha} m_{\alpha}(\delta_{ij} r_{\alpha}{}^{2} - r_{\alpha i} r_{\alpha j})$ with the notation used is standard.

4a.) Define principal axis of inertia for a rigid body and show that there are three of them which are either orthogonal to each other or can be chosen to be orthogonal.

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Give an example to show that the principal axes of inertia may not be unique.

4b.) Write, without proof, the generic form of the moment of inertia matrix if the coordinate axes are along the principal axes of inertia.

4c.) Show that if the rigid body has an axis of symmetry, then the axis of symmetry is always one of the principle axes of rotation.

For the next parts of the question suppose that the rigid body is a uniform cube (of mass *M* and with each side of length *b*).

4d.) If the cube is rotating with one of its corner fixed, identify one of the principle axes of inertia in this case.

4e.) If the cube is rotating around its centre of mass (CM) determine all components of the moment of inertia matrix I_{ij} with the choice of CM as the origin and the coordinate axes parallel to the edges of the cube)

4f.) Show that in part d.) the moment of inertia matrix I_{ij} remains the same even if the coordinate axes are not parallel to the edges of the cube. (Hint: Show that the moment of inertia matrix I and I' in two coordinate systems $\vec{x}, \vec{x}'$ is related by $I' = R^T I R$ where R is the rotation matrix connecting $\vec{x}, \vec{x}'$; i.e. $x'_i = \sum_j R_{ij} x_j$)

Do EITHER Q5 OR Q6 below.

Q5. [Total Marks: 3+5+4=12]

Consider a pendulum of length l and a bob of mass m at its end moving through oil with so that θ, its angle with the vertical, decreasing. The bob undergoes small oscillations but the oil retards the bob's motion with a resistive force $2m\sqrt{\frac{g}{l}}(l\dot{\theta})$. The bob is initially pulled back at t = 0 with $\theta = \alpha, \dot{\theta} = 0$

5a.) Set up the equation of motion for small values of θ.
5b.) Find the angular speed as a function of time.
5c.) Show that in this case the pendulum will never swing to the other side.

Q6. [Total Marks: 7+5=12]

A projectile is fired at an angle of 45° with initial kinetic energy E_0. At the top of its trajectory, the projectile explodes with additional energy E_0 into two fragments. One fragment of mass m_1 travels straight down.

6a.) What is the velocity of the first fragment and the vertical component of the velocity second fragment of mass m_2 immediately after the explosion?
6b.) What the maximum value for the ratio m_1 / m_2 in this problem?

Figure 1 for Question 3.